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# Determining the Stability Criteria for the Suggested Nonlinear Autoregressive Model

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**Abstract**

We present in this current study the stability properties of the model, which primarily relies on methods derived from Ozaki's work. The proposed mathematical simplification method makes nonlinear model analysis more achievable through an approximate solution. Stability conditions for the proposed model's behavior throughout time represent the main focus of this research investigation. Many scientific disciplines, including economics and biology, implement nonlinear models broadly because these models offer effective solutions to complex dynamical systems, which regular techniques find difficult to handle, along with oscillatory and chaotic phenomena. The analysis of these systems becomes simpler through localization at the non-zero point using Ozaki's model, which allows the use of standard linear stability analysis methods. Our investigation targets the singular point while developing conditions that describe its stability range and determining the stability rule for newly expected limit cycles. The theoretical findings undergo verification through different numerical examples, which adhere to derivative specifications .

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# تحديد معايير الاستقرار لنموذج الانحدار الذاتي غير الخطي المقترح

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## المستخلص

تظهر دراستنا الحالية خواص استقراريه النموذج المقترح المعتمد أساسا على تقنيات العالم اوزاكي. من خلال النهج الرياضي التقريبي لتبسيط تحليل النماذج الغير خطية. هدفنا الأساسي من هذا البحث هو ابراز شروط استقراريه سلوك النموذج المقترح بمرور الزمن. اعتمد النموذج الغير خطي بشكل واسع في العديد من التخصصات العلمية مثل علم الاقتصاد والاحياء وذلك لا مكانيته العالية وكفاءته في التعامل مع الأنظمة والنماذج الديناميكية المعقدة التي يصعب التعامل معها بالتقنيات التقليدية المعروفة ومنها الأنظمة الحركية المتذبذبة والأنظمة الديناميكية الفوضوية. ولتبسيط تحليل هذه الأنظمة يكون النموذج محليا حول النقطة الغير صفرية وفقا لنموذج العالم اوزاكي ليتيح لنا تطبيق تقنية تحليل الاستقرار الخطي. يركز بحثنا على تحديد النقطة المنفردة ووضع الشروط التي تحدد مجال استقرارها إضافة الى شرط الاستقرار لأي دورة حدية جديدة متوقعة. يتم التأكد من صحة النتائج النظرية باستخدام امثلة عددية متنوعة تتوافق مع المعايير المشتقة وبكل الأحوال توفر النماذج المقترحة إطار استقرار متين لأي بحث مستقبلي في المجالات العلمية المماثلة.

**الكلمات المفتاحية:** نموذج أوزاكي؛ شروط الاستقرار ؛ نماذج الانحدار.

# 1. Introduction

Many previous researches and studies have used Ozaki's technique to study and find the conditions for the stability of discrete-time nonlinear models. Researchers Salim and Youns conducted a study on the nonlinear model as presented in(A. Salim & Abdullah, 2014). Youns and Salim found the stability for a nonlinear model that contains the hyperbolic triangle function in(A. S. Youns & Salim, 2018). In (A. S. Y. Youns & Salim, 2019), Youns and Salim examined a novel formulation of a “nonlinear time series model” incorporating the hyperbolic secant function. In (A. S. Youns, 2019), Youns developed a “nonlinear model incorporating a fractional function”. Youns and Ahmad identified the stability of the nonlinear time series model in (A. S. Youns & Ahmad, 2023). A "Nonlinear Autoregressive Model and found the Stability conditions with Prediction" was proposed by Ahmad, Youns, and Hamdi in (Ahmad et al., 2025). The "stability of exponential (GARCH) models" was examined by Muhammad and Mudhir in (Mohammad & Mudhir, 2020). The "orbital stability of the log-logarithmic autoregressive model with application" was proposed by Hamad and Mohammad in (Hamad & Mohammad, 2025). Mohammad and Salim outlined a particular instance of the "logistic autoregressive model," a polynomial autoregressive model in (Mohammad & Salim, 2007)."Studying the stability by using local linearization method" was the topic of Salim and Ebrahem's work in (A. J. Salim et al., 2020). "Non-linear time series models and dynamical systems" was an offer made by Ozaki in (Ozaki, 1985). "The statistical analysis of perturbed limit cycle processes using nonlinear time series models" was put forward by Ozaki in (Ozaki, 1982). The "stability conditions of the Burr X autoregressive model" were examined by Khalaf and Mohammad in (Khalaf & Mohammad, 2019). The " Stability requirements of the Pareto AR model " were discovered by Hamdi, Mohammad, and Khaleel in (Hamdi et al., 2018).

Based on the above, this study focuses on establishing stability criteria for a proposed nonlinear model. Using a local linearization approach, the research aims to identify the singular points and their associated stability conditions, as well as the limit cycle stability conditions of the proposed model.

## 2. The Suggest Model:

The nonlinear suggest time series model is that:

$$w_t = \sum_{i=1}^{p}[c_i + d_i(e^{-2w_{t-1}+1})]\, w_{t-i} + \varepsilon_t; w_{t-1} \neq 0 \qquad (1)$$

where, $c_1, \ldots, c_p; d_1, \ldots, d_p$ indicate a constant, $\{\varepsilon_t\}$is random noise.

## 3. Technical derivation for the proposed model:

Ozaki's local linear approximation was applied to the proposed model, and stability requirements were determined.

### 3.1 Identify *Z* (singular point):

Consider the proposed first-order model, such as

$$w_t = [c_1 + d_1(e^{-2w_{t-1}+1})]w_{t-1} + \varepsilon_t; w_{t-1} \neq 0 \qquad (2)$$

Hence,$Z = f(Z)$, and the random noise $\varepsilon_t = 0$ in (2). Then

$$Z = [c_1 + d_1(e^{-2Z+1})].Z$$

Since,$(Z \neq 0)$, $(d_1 \neq 0)$ and $(c_1 \neq 1)$.

Therefore

$$Z = -\frac{1}{2}(log(\frac{1-c_1}{d_1}) - 1) \qquad (3)$$

Where,

$$1 - c_1 \neq 0, d_1 \neq 0 .$$

The suggested model of order two

$$w_t = [c_1 + d_1(e^{-2w_{t-1}+1})]w_{t-1} + [c_2 + d_2(e^{-2w_{t-1}+1})]w_{t-2} + \varepsilon_t; w_{t-1} \neq 0 \qquad (4)$$

Whenever.

$$Z = [c_1 + d_1(e^{-2Z+1})]Z + [c_2 + d_2(e^{-2Z+1})]Z$$

Therefore,

$$Z = -\frac{1}{2}(log(\frac{1-\sum_{i=1}^{2} c_i}{\sum_{j=1}^{2} d_j}) - 1) \qquad (5)$$

Where,

$1-\sum_{i=1}^{2} c_i \neq 0; (\sum_{j=1}^{2} d_j) \neq 0$ .

Equation (1)'s singular unique point was determined by using a method similar to that of equations (3) and (5), so that

$$Z = -\frac{1}{2}(log(\frac{1-\sum_{i=1}^{p} c_i}{\sum_{j=1}^{p} d_j}) - 1) \qquad (6)$$

Where,

$(1-\sum_{i=1}^{p} c_i) \neq 0; (\sum_{j=1}^{p} d_j) \neq 0$

.

## 3.2 Singular point stability:

put $w_s = Z + Z_s$ , for all $s = t; t-1$ ; $\varepsilon_t = 0$ in (2).

Since, $Z_s; s = t, t-1$ is exceedingly small, then $Z_t . Z_{t-1} = 0$.

Also, $\forall s = t; t-1; \forall n \geq 2, Z_s^n \to 0$

Hence, by applying the Taylor series expansion in (2), therefore

$$Z + Z_t = [c_1 + d_1(e^{-2(Z+Z_{t-1})+1})](Z + Z_{t-1}) \qquad (7)$$

To obtain

$$Z_t = [c_1 + d_1(1-2Z)e^{-2Z+1}]Z_{t-1} \qquad (8)$$

Or,

$$Z_t = [\frac{c_1 d_1 + (1-c_1) d_1 \, log(\frac{1-c_1}{d_1})}{d_1}]Z_{t-1} \qquad (8)$$

Therefore

$$Z_t = a_1 Z_{t-1}; a_1 = \frac{c_1 d_1 + (1-c_1) d_1 \, log(\frac{1-c_1}{d_1})}{d_1} \qquad (9)$$

Therefore, if the root of equation (8) lies in the unit circle, equation (9) represents a stable first-order model.

$|r_1| = |a_1| < 1$ .

The stability of the single point of equation (4) can be determined in a similar manner to the stability condition for equation (2) above.

Therefore

$$Z + Z_t = [c_1 + d_1(e^{-2(Z+Z_{t-1})+1})](Z + Z_{t-1}) + [c_2 + d_2(e^{-2(Z+Z_{t-1})+1})](Z + Z_{t-2}) \quad (10)$$

Then

$$Z_t = [c_1 + d_1(1 - 2Z)e^{-2Z+1} - 2d_2e^{-2Z+1}Z]Z_{t-1} + [c_2 + d_2e^{-2Z+1}]Z_{t-2}$$

$$Z_t = a_1Z_{t-1} + a_2Z_{t-2} \quad (11)$$

Where,

$$a_1 = [c_1 + d_1(1 - 2Z)e^{-2Z+1} - 2d_2e^{-2Z+1}Z]_1; a_2 = [c_2 + d_2e^{-2Z+1}]$$

The distinguishing equation

$$(v - r_1).(v - r_2) = v^2 - a_1(v) - a_2 = 0 .$$

Therefore,

$$a_1 = (r_1 + r_2), a_2 = -r_1r_2.$$

Therefore, the roots of $v^2 - a_1v - a_2 = 0$ are $r_1, r_2$ .

Then, the condition that is stationary

$$(|r_i|) < 1 \quad i = 1 ; 2 .$$

The single stability condition of equation (1) is

$$Z_t = [c_1 + d_1(1 - 2Z)e^{-2Z+1} - \sum_{j=2}^{p} 2d_je^{-2Z+1}Z]Z_{t-1} + \sum_{j=2}^{p}[c_j + d_je^{-2Z+1}]\, Z_{t-j}$$

## 3.3 The limit cycle

A period q limit cycle of $w_t = w_t; w_{t+1}; w_{t+2}; \ldots; w_{t+q}$, for suggested model in the equation (2).

When $w_s$ is a point nearly a limit cycle is replaced

$$\forall s = t, t - 1; w_s = w_s + Z_s \;, \varepsilon_t = 0.$$

$$w_t + Z_t = [c_1 + d_1(e^{-2(w_{t-1}+Z_{t-1})+1})](w_{t-1} + Z_{t-1}) \quad (12)$$

Then

$$Z_t = [c_1 + d_1 e^{-2w_{t-1}+1}(1 - 2w_{t-1})]Z_{t-1} \quad (13)$$

Let, t=t+q in (13)

$$Z_{t+q} = [c_1 + d_1 e^{-2w_{t+q-1}+1}(1 - 2w_{t+q-1})]Z_{t+q-1} \quad (14)$$

Therefore

$$Z_{t+q} = \prod_{i=1}^{q}[c_1 + d_1 e^{-2w_{t+q-i}+1}(1 - 2w_{t+q-i})]Z_t \quad (15)$$

Then

$$\left|\frac{Z_{t+q}}{Z_t}\right| = \left(\left|\prod_{(i=1)}^{q}[c_1 + d_1 e^{-2w_{t+q-i}+1}(1 - 2w_{t+q-i})]\right|\right) < 1 \quad (16)$$

Then, (16)

$$\left|\frac{Z_{t+q}}{Z_t}\right| = \left|\prod_{i=1}^{q}[c_1 + d_1 e^{-2w_{t+i-1}+1}(1 - 2w_{t+i-1})]\right| < 1 \quad (17)$$

In the equation (4). the formula $w_t = w_t; \ldots; w_{t+q}$ , $\varepsilon_t = 0$.

Let, $w_s = w_s + Z_s, \forall s = t; t-1; t-2$, to reach that:

$$w_t + Z_t = [c_1 + d_1(e^{-2(w_{t-1}+Z_{t-1})+1})](w_{t-1} + Z_{t-1}) + [c_2 + d_2(e^{-2(w_{t-1}+Z_{t-1})+1})](w_{t-2} + Z_{t-2}) \quad (18)$$

Therefore

$$Z_t = [c_1 + d_1 e^{-2w_{t-1}+1}(1 - 2w_{t-1} - 2w_{t-2})]Z_{t-1} + [c_2 + d_2(e^{-2w_{t-1}+1})]Z_{t-2} \quad (19)$$

Put in (19), t=t+q , to reaches

$$Z_{t+q} = [c_1 + d_1 e^{-2w_{t+q-1}+1}(1 - 2w_{t+q-1} - 2w_{t+q-2})]Z_{t+q-1} + [c_2 + d_2(e^{-2w_{t+q-1}+1})]Z_{t+q-2} \quad (20)$$

Then, (20). is equal to

$$Z_{t+q} = \prod_{i=1}^{q}[c_1 + d_1 e^{-2w_{t+q-i}+1}(1 - 2w_{t+q-i} - 2w_{t+q-(i+1)})]Z_t + \prod_{i=1}^{q-1}[c_2 + d_2 e^{-2w_{t+q-i}+1}]Z_t \quad (21)$$

Equation (21). is orbital stable when the stationary condition satisfied that:

$$\left|\frac{Z_{t+q}}{Z_t}\right| = \left|\prod_{i=1}^{q}[c_1 + d_1 e^{-2w_{t+i-1}+1}(1 - 2w_{t+i-1} - 2w_{t+i-2})]Z_t + \prod_{i=2}^{q}[c_2 + d_2 e^{-2w_{t+i-1}+1}]Z_t\right| < 1 \quad (22)$$

## 4. Numerical Examples

The examples in this paragraph demonstrate to identify a point for the suggested first-order model to verify the Special criteria for both the stabilization of a point, the stability of any potential limit cycle. In Example 1, the singular point is identified and meets the stability condition. Using Matlab, we plot the model's trajectories with different initial values, showing that the model stabilizes at the singular point, as illustrated in Figure 1. Hence, the model is orbitally stable with no limit cycle. In Example 2, the singular point was found but does not satisfy the stability condition. Matlab is again used to plot the trajectories with various initial values, revealing that the model does not stabilize at the singular point, as shown in Figure 2. Consequently, the model is not orbitally stable but meets the limit-cycle stability condition for a cycle of length q=2.

## Example (1):

If, $c_1 = 0.9, d_1 = 0.2, \varepsilon_t = 0$ , in (2) to reach

$$w_t = [0.9 + 0.2(e^{-2w_{t-1}+1})]w_{t-1}; w_{t-1} \neq 0$$

Then, by using equation (3) to reaches that

$$Z = -\frac{1}{2}(log(\frac{c_1 - 1}{d_1}) - 1) = 0.8466$$

Hence; by used (9), then $Z_t = 0.8303Z_{t-1}$ for $Z = 0.8466$ .

In this case, the singular (stability) point is shown to be stable since a root for preceding equation lies in a unity circle. The stability of the model is shown in Figure 1 below for a range of beginning values.

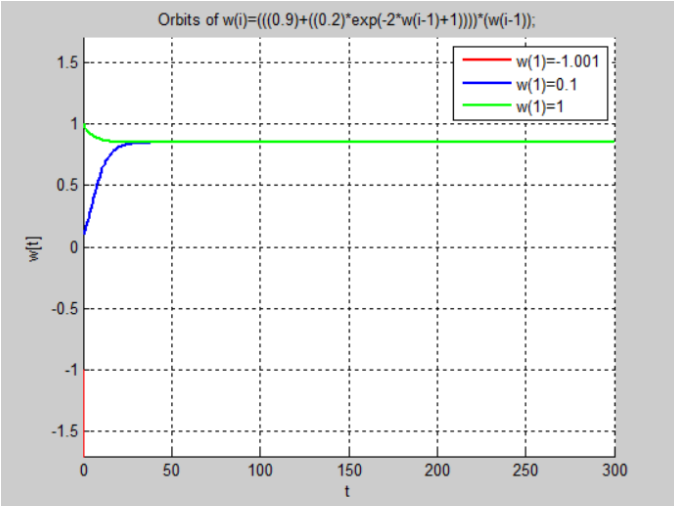


**Figure 1:** Z = 0.8466 is a stable singular point with various initial values.

## Example (2):

Whenever, $c_1 = -0.3, d_1 = 2.5, \varepsilon_t = 0$ , in (2) to reach $w_t = [-0.3 + (2.5)(e^{-2w_{t-1}+1})]w_{t-1}; w_{t-1} \neq 0$

Then, by using equation (3) to get that

$$Z = (\frac{d_1}{c_1 - 1}) == (\frac{-1.6}{-1.059 - 1}) = (\frac{-1.6}{-2.059}) = 0.77$$

Since, $Z = 0.097$ , and used (9), to found $Z_t = (-1.059)Z_{t-1}$ .

$z$ is unstable because the root of the previous equation is located outside the unit circle. The instability of the model with different initial values is shown in Figure (2) below.

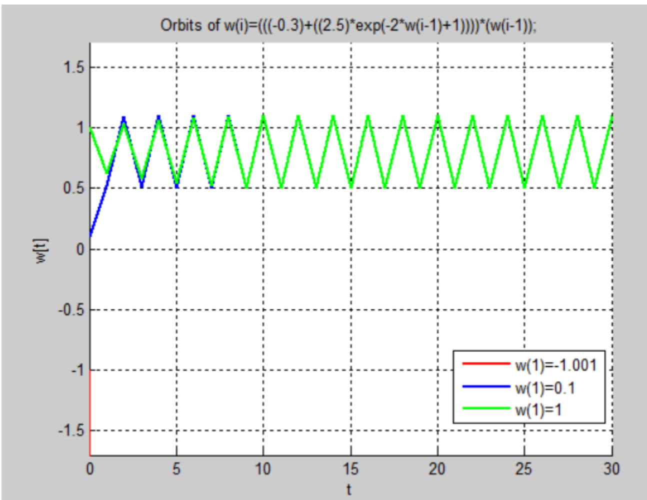


**Figure (2).** Z=0.77 is a not stable singular point with various initial values.

The limit cycle of period q=2 which is $\{x_0 = 1.1, x_1 = 0.5, x_2 = 1.1 = x_0\}$.

Then, by used (16) we get

$$\left|\frac{z_{t+2}}{z_t}\right| = \left|\prod_{i=1}^{2}[-0.3 + 2.5e^{-2w_{t+2-i}+1}(1 - 2w_{t+2-i})]\right| = |[-1.2036][-0.3]| = |0.3611| < 1$$

Then, the limit cycle is stable.

## 6. Conclusions

1- The study introduces a nonlinear autoregressive framework made up of a linear component alongside a nonlinear term (expressed via a specific nonlinear function), and it derives the criteria ensuring both singular-point stability and the existence of a limit cycle for this model.
2- In Example 1 of the study, we present a numerical case that fulfills the proposed model’s singular-point stability requirements.
3- In Example 2 of the research, we demonstrate a numerical instance that meets the limit-cycle stability criteria for the model.

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