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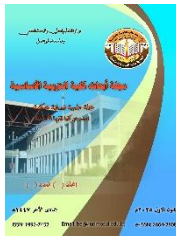

Banach Contraction Method to Solve Partial Integral-differential Equations

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Article Information

Article history:
Received: August 24,2024
Reviewer: January 1, 2025
Accepted: January 5, 2025
Available online

Keywords:
Banach Contraction Method (BCM), Partial Integro-differential equations (PIDF)

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Abstract

In this paper, Banach contraction method was used to solve linear and nonlinear partial integra differential equations of Volterra type and compared them with exact solutions to demonstrate accuracy of the proposed method. Results revealed that Banach contraction method is very effective, simple and of a high accuracy to solve higher order Integro -differential equations. Four different examples were solved and compared with exact solutions using mean squared error and absolute error. The tables and figures show convergence of proposed method in solution.

P-ISSN: 1992–7452 **E-ISSN: 2664-2808**

طريقة انكماش باناخ لحل المعادلات التفاضلية التكاملية الجزئية

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المستخلص

في هذا البحث، تم استخدام طريقة انكماش باناخ لحل المعادلات التفاضلية التكاملية الجزئية الخطية وغير الخطية من نوع فولتيرا وفريدهولم ومقارنتها بالحلول الدقيقة لإثبات دقة الطريقة المقترحة. كشفت النتائج أن طريقة انكماش باناخ فعالة للغاية وبسيطة وذات دقة عالية لحل المعادلات التفاضلية التكاملية من الرتب العليا. تم حل أربعة أمثلة مختلفة ومقارنتها بالحلول الدقيقة باستخدام متوسط الخطأ التربيعي والخطأ المطلق. تظهر الجداول والأشكال تقارب الطريقة المقترحة في الحل.

الكلمات المفتاحية: طريقة انكماش باناخ (BCM)، المعادلات التكاملية التفاضلية الجزئية (PIDF).

1. Introduction

Partial integral differential equations play an important role in many fields of science and engineering [1] . Recently , researchers have paid attention to solving partial integro- differential equations due to their many applications in all branches of science, especially in physical phenomena .These equations originate in biological fluid dynamics Chemical and kinetic models [2,3]. Various numerical schemes are proposed by Dhegihan to solve PIDEs arising in viscoelasticity[4]. PIDEs have been used in jump-diffusion models for pricing of derivatives in finance[5]. Abergel used a nonlinear PIDE in financial modelling [6]. Hepperger proposed a PIDE in the model of electricity swaptions[7]. A PIDE governing biofluid flow in fractured biomaterials is proposed by Zadeh in[8]. There are several methods for solving Integro -differential equations, in (1988) E. G. Yanik and G. Fairweather use finite element methods for solving Integral -differential equation of parabolic type [9]. In (1989) M. N. Leroux and V. Thomèe use Numerical solution of semi linear Integral differential equations of parabolic type with non-smooth data. The stability of Ritz-Volterra projections and error estimates for finite element methods for a class of Integro -differential equations of parabolic type is studied by Y. Lin and T. Zhang [10]. In (1992), A. K. Pani, V. Thomèe, and L.B. Wahlbin use Numerical methods for hyperbolic and parabolic integro -differential equations [11]. Global and blow-up solutions of a class of semi linear Integro - differential equation, by Cui Shang-bin and Ma Yu-lan in (1994) [12]. I. H. Sloan and V. Thomèe, use Time discretization of an Integro -differential equation of parabolic type [13]. Integral- differential equations are usually difficult to solve with analytical methods, we resort to using the approximate solution to obtain an effective solution[14] ,such as the pseudo spectral method [15] , variation iteration Method [16] , Adomian decomposition method[17]…etc. In this research, a well-known iterative method called the Banach contraction method was used to solve linear and nonlinear partial differential equations of the Volterra type. Banach's fixed point theory is also called the contraction map principle and was mentioned for the first time in 1922 where the theory was named after Stefan Banach (1892-1945)[18]. The BCM characterized as one of the developments of Picard's method where the ease of application clearly observed which makes it distinct from the other known iterative methods Banach's fixed point theorem was expanded by Nadler from single-valued maps to set-valued contractive maps[19] There are some different types of partial Integral -differential equations, such as hyperbolic type partial differential equations[20] Parabolic Partial Integro-Differential Equation [21] etc... .

In this paper, We will solve the partial integro differential equations by used Banach contraction which have general form[22]:

$$y_t(x,t) + \alpha_1 y_{xx}(x,t) = \alpha_2 \int_0^t r_1\big(x,t,s,y(x,s)\big)\,\mathrm{ds} + \alpha_3 \int_0^T r_2\big(x,t,s,y(x,s)\big)\,\mathrm{ds} + f(x,t) \qquad (1)$$

with the initial and boundary conditions as:

$$y(x,0) = w_0(x) \quad y(0,t) = w_1(t), \; y(b,t) = w_2(t) \qquad (2)$$

Where $x \in [0,b]$, $\mathrm{t} \in [0,T]$, α_1, α_2 and α_3 are constants and the functions $f(x,t), r_1\big(x,t,s,y(x,s)\big)$, $r_2\big(x,t,s,y(x,s)\big)$ are supposed to sufficiently smooth on $\mathrm{J} := [0,b] \times [0,T]$

and D where D:=$\{(x,t,s): x \in [0,b], s,t \in [0,T]\}$, any function $y(x,s)$ define over

$[0,b] \times [0,T]$

The paper is arranged as follows. In the second section, basic definitions related to research are presented. In Section 3, the basic concepts of the BCM are presented. In Section 4, The Proposed technique BCM. In Section 5, examples are presented with results. In Section 6, the conclusions are presented**.**

2. Basic Concepts

In this part, some basic definitions and concepts related to the research topic will be presented

Definition 2. 1 [23]: The Maximum Absolute Error (MAE) : It is the absolute value of the difference between the exact solution and the approximate solution, as follows:

$$\left\|z_{Exact}(t) - \Phi_q(t)\right\|_\infty = \max_{a \le x \le b}\left\{\left|z_{Exact}(t) - \Phi_q(t)\right|\right\} \qquad (3)$$

Definition 2. 2 [23]: Mean Square Error (MSE)

It is the sum of the square of the difference between the exact solution $Ex(x_j)$ and the approximate solutions $\emptyset(x_\mathrm{j})$ divided by

The number of points that were used is M and as in the following formula:

$$MSE = \frac{\sum_{\tau=1}^{M}\left(E\ t\ (t_\tau) - \Phi(t_\tau)\right)^2}{M} \quad (4)$$

Where $\vec{t}_\tau$ are vectors since $\tau = 1,2, \ldots$

3. Banach contraction method [24]

Let us obtain the following nonlinear functional equation:

$$L(v(x)) = N(v(x)) + g\ (x) = 0 \qquad 0 \le x \le 1 \quad (5)$$

With the initial condition

$$v(0) = b \quad (6)$$

Where x represents the independent variable , $v(x)$ is the unknown function, $g\ (x)$is a given known function, $L = \frac{d}{dx}$ is the linear operator, N is the nonlinear operator, b is constant

By adding the operator L^{-1} which is the inverse of L to equation (5), we get

$$v(x) = v(0) + L^{-1}\left(N(v(x)) + g\ (x)\right) \quad (7)$$

The solution y for Eq. (7) can be given by the following series

$$v(x) = \sum_{c=0}^{\infty} v_c\ (x) \quad (8)$$

With compensation Eq. (8) in (7) ,we get:

$$\sum_{c=0}^{\infty} v_c\ (x) = d(x) + L^{-1}\left(N\left(v \sum_{c=0}^{\infty} v_c\ (x)(x)\right)\right) \quad (9)$$

Where

$$d(x) = v(0) + L^{-1}(g\ (x)) \quad (10)$$

We get the successive iterative approximations as following:

$$v_0 = d(x)$$

$$v_1 = v_0 + L^{-1}(N(v_0))$$

$$v_2 = v_0 + L^{-1}(N(v_1))$$

$$\vdots$$

$$v_q = v_0 + L^{-1}(N(v_0)), \quad q = 1,2,\cdots, \quad (11)$$

This process can be continued to obtain the n^{th}approximation. The general form of BCM can be written as:

$$v = \lim_{q \to \infty} v_q \quad (12)$$

4.The convergence of the Banach contraction method[25]

In this section, some basic concepts and theories are presented according to the convergence principles of the Banach contraction method ; to prove the convergence method (BCM), the following iterations must be applied to equation (5) with initial condition (6) , where the convergence iterations are formulated as follows:

$z_0 = v_0$

$z_1 = M[z_0]$

$z_2 = M[z_0 + z_1]$

$z_3 = M[z_0 + z_1 + z_2]$

$\vdots$

$$z_{q+1} = M\left[z_0 + z_1 + z_2 + \cdots + +z_q\right] \quad (13)$$

Where M represents the following operator

$$M\left[z_q\right] = \Psi_k - \sum_{c=0}^{k-1} z_c\,(x) \quad k \geq 1 \quad (14)$$

Where , Ψ_k is represents the solution to the problem and is in the form , for the BCM:

$$\Psi_k = z_0 + L^{-1}\left(N\left(\sum_{c=0}^{k-1} z_c\,(x)\right)\right) \qquad k \geq 1 \quad (15)$$

Therefore, using the equations (13), (14) and (15) , we obtain the approximate solution sequentially as follows :

$$\mathrm{v}(x) = \lim_{q\to\infty} \mathrm{v}_q = \sum_{c=0}^{\infty} z_c\,(x) \quad (16)$$

Theorem4.1. Let $M: H \to H$ be an operator ,where H define as Hilbert space, if there exist

$0 < \varphi < 1$ s.t $M[z_0 + z_1 + z_2 + \cdots + +z_{c+1}] \leq \varphi M[z_0 + z_1 + z_2 + \cdots + +z_c]$ where

$z_{c+1} \leq z_c \;\; \forall c = 0,1,2,..$

Then the series solution converges $\mathrm{v}_q(x) = \sum_{c=0}^{q} z_c\,(x)$ .Fulfilling the condition of this theory is sufficient to achieve convergence. This theory represents a special case of Banach's fixed point theory.

Proof. See[25].

Theorem4.2. the series solution $\mathrm{v}(x)=\sum_{c=0}^{\infty} z_c\,(x)$ will represent the exact solution of the problem ,if this series is convergent

Proof. See[25].

Theorems 4.1 and 4.2 state that the equation (13), which represents the approximate solution by the Banach contraction method you are getting closer to the exact solution According to the condition$\ni 0<\varphi<1$ s.t $M[z_0+z_1+z_2+\cdots++z_{c+1}]\leq \varphi M[z_0+z_1+z_2+\cdots++z_c]$

where $z_{c+1}\leq z_c\ \ \forall c=0,1,2,..$ In other words, if we have the parameter

$$\lambda_c=\begin{cases}\frac{\|z_{c+1}\|}{\|z_c\|} & \|z_c\|\neq 0\\ 0 & \|z_c\|=0\end{cases} \qquad (17)$$

Then the series solution $\sum_{c=0}^{\infty} z_c\,(x)$ for the nonlinear ODE given by (5) will be convergent to the exact solution $\mathrm{v}(x)$ by using Banach contraction method if the condition is met $0\leq\lambda_c<1$,

$\forall c=0,1,2,\ldots\ x,t\in[0,1]$

5. The Proposed technique BCM

For solving the equation (1) using Banach contraction Method, with the initial conditions :

$$y(x,0)=\ w_0(x) \qquad (18)$$

By adding the operator $\mathcal{L}_t^{-1}$ which is the inverse of $\mathcal{L}_t$ to equation (1), we get

$$y(x,t)=w_0(x)+\ \mathcal{L}_t^{-1}\Big(\alpha_2\int_0^t r_1\big(x,t,s,y(x,s)\big)\,\mathrm{ds}+\alpha_3\int_0^T r_2\big(x,t,s,y(x,s)\big)\,\mathrm{ds}-\alpha_1 y_{xx}(x,t)+f(x,t)\Big) \qquad (19)$$

The solution y for Eq. (19) can be given by the following series (Daftardar-GejjiandJafari,2006)

$$y(x,t)=\sum_{c=0}^{\infty} y_c\,(x,t) \qquad (20)$$

$$y(x,s)=\sum_{c=0}^{\infty} y_c\,(x,s) \qquad (21)$$

With compensation Eq. (20) and (21) in (19) ,we get:

$$\sum_{c=0}^{\infty} y_c(x,t) = m(x,t) + \mathcal{L}_t^{-1}\Big(\alpha_2 \int_0^t r_1\big(x,t,s,\sum_{c=0}^{\infty} y_c(x,s)\big)\,ds + \alpha_3 \int_0^T r_2\big(x,t,s,\sum_{c=0}^{\infty} y_c(x,t)\big)\,ds - \alpha_1 y_{xx}(x,t)\Big) \quad (22)$$

Where

$$m(x,t) = w_0(x) + \mathcal{L}_t^{-1}\big(f(x,t)\big) \quad (23)$$

We get the successive iterative approximations as following:

$y_0(x,t) = m(x,t),$

$y_1(x,t) = y_0 + \mathcal{L}_t^{-1}\Big((\alpha_2 \int_0^t r_1\big(x,t,s,y_0(x,s)\big)\,ds + \alpha_3 \int_0^T r_2\big(x,t,s,y_0(x,s)\big)\,ds - \alpha_1 y_{0_{xx}}(x,t)\Big)$

$y_2(x,t) = y_0 + \mathcal{L}_t^{-1}\Big((\alpha_2 \int_0^t r_1\big(x,t,s,y_1(x,s)\big)\,ds + \alpha_3 \int_0^T r_2\big(x,t,s,y_1(x,s)\big)\,ds - \alpha_1 y_{1_{xx}}(x,t)\Big)$

$\vdots$

$$y_q(x,t) = y_0 + \mathcal{L}_t^{-1}\Big((\alpha_2 \int_0^t r_1\big(x,t,s,y_{q-1}(x,s)\big)\,ds + \alpha_3 \int_0^T r_2\big(x,t,s,y_{q-1}(x,s)\big)\,ds - \alpha_1 y_{q-1_{xx}}(x,t)\Big), \quad q = 1,2,\cdots \quad (24)$$

And the approximate solution will be given in

$$y(x,t) = \lim_{q\to\infty} y_q \quad (25)$$

6. Illustrative Examples

In this section, MAPLE program used to resolve different examples of multi-divisions limits in equation (1).

Example 1[26]:

Let us have the following linear partial Integro-differential equations:

$$y_t(x,t) - y_{xx}(x,t) = (2t - x^2 - t^2 x)e^{-xt} + \frac{x(e^{-t} - e^{-xt})}{x-1} - \int_0^t e^{s-t}\, y(x,s)\,ds \quad (26)$$

with the initial and boundary conditions as:

$$y(x,0) = x \quad y(0,t) = 0, \; y(1,t) = e^{-t} \qquad x,t \in [0,1] \quad (27)$$

The exact solution is $z_{Exact}(x,t) = xe^{-xt}$.

Using the technique described in Part 5we will obtain the following iterations:

$$y_0 = x - \frac{x^2}{x-1} + \frac{2x}{x-1} - \frac{1}{x-1} + \frac{x^2}{(x-1)e^{xt}} - \frac{x}{(x-1)e^{xt}} + \frac{xt^2}{(x-1)e^{xt}} + \cdots$$

$$y_1 = x + \frac{2x}{x-1} - \frac{1}{x-1} + \frac{x^2}{(x-1)e^{xt}} - \frac{x}{(x-1)e^{xt}} + \frac{xt^2}{(x-1)e^{xt}} + \frac{4xt^2}{(x-1)^3e^{xt}} + \cdots$$

$$y_2 = x + \frac{2x}{(x-1)} - \frac{1}{(x-1)} + \frac{xt^2}{(x-1)e^{xt}} + \frac{x^2}{(x-1)e^{xt}} + \frac{x}{(x-1)e^{xt}} - \frac{t^2}{(x-1)e^{xt}} + \cdots$$

$$y_3 = x - \frac{1}{x-1} + \frac{2x}{(x-1)} + \frac{xt^2}{(x-1)e^{xt}} - \frac{2946\,xt^5}{(x-1)^3(x^4-4x^3+6x^2-4x+1)e^{xt}} - \frac{20280t^4}{(x-1)^3(x^4-4x^3+6x^2-4x+1)e^{xt}} + \cdots$$

To prove the convergence of the approximate iterations using Banach's method from the exact solution, we will use the iterative formulas in the equation (13) with the equations (14),(15).

Where in eq(14) M define as :

$$M[z_q] = \Psi_k - \sum_{c=0}^{k-1} z_c(x,t) \quad k \geq 1 \qquad (28)$$

Where

$$\Psi_k = z_0 + \mathcal{L}_t^{-1}\left(\left(\sum_{c=0}^{k-1} z_c(x,t)\right)_{xx}(x,t) - \int_0^t e^{s-t} \sum_{c=0}^{k-1} z_c(x,s)\mathrm{ds}\right) \qquad k \geq 1 \qquad (29)$$

Then , Substituting the equation(28) and(29) into (13),we get the following iterative formulas:

$$z_0(x,t) = y_0 = x - \frac{x^2}{x-1} + \frac{2x}{x-1} - \frac{1}{x-1} + \frac{x^2}{(x-1)e^{xt}} - \frac{x}{(x-1)e^{xt}} + \frac{xt^2}{(x-1)e^{xt}} + \cdots$$

$$z_1(x,t) = z_0(x,t) + z_{0_{xx}}(x,t) - \left(\int_0^t e^{s-t} z_0(x,s)\mathrm{ds}\right) - z_0(x,t)$$

$$= \frac{4xt^2}{e^{xt}(x-1)^3} - \frac{2}{(x-1)^3} - \frac{9t^2}{e^{xt}(x-1)^3} + \frac{18}{x^2(x-1)^3} + \frac{6}{x(x-1)^3} + \cdots$$

$$z_2(x,t) = z_0(x,t) + (z_0+z_1)_{xx}(x,t) - \left(\int_0^t e^{s-t}(z_0+z_1)(x,s)\mathrm{ds}\right) - (z_0+z_1)$$

$$= \frac{4xt^2}{e^{xt}(x-1)^3} + \frac{2}{(x-1)^3} + \frac{9t^2}{e^{xt}(x-1)^3} - \frac{18}{x^2(x-1)^3} - \frac{6}{x(x-1)^3}$$

$$z_3(x,t) = z_0(x,t) + (z_0+z_1+z_2)_{xx}(x,t) - \left(\int_0^t e^{s-t}(z_0+z_1+z_2)(x,s)\mathrm{ds}\right) - (z_0+z_1+z_2)$$

$$= \frac{-217xt^2}{(x-1)^3(x^4-4x^3+6x^2-4x+1)} + \frac{262880t}{x^{10}(x-1)^3(x^4-4x^3+6x^2-4x+1)} - \frac{2488166}{x^4(x^7-7x^6+21x^5-35x^4+35x^3-21x^2+7x-1)}$$

According to paragraph 4 with theorems 4.1 and 4.2, the approximate series is a convergence condition if the values $\lambda_c < 1, \forall c \geq 0$. By applying the equation (17) we get the values of λ_c as follows:

$$\lambda_0 = \frac{\|z_1(x,t)\|}{\|z_0(x,t)\|} = 0.02442 < 1$$

$$\lambda_1 = \frac{\|z_2(x,t)\|}{\|z_1(x,t)\|} = 0.00134 < 1$$

$$\lambda_2 = \frac{\|z_3(x,t)\|}{\|z_2(x,t)\|} = 0.00046 < 1$$

Since all values of λ_c are less than one $\forall c = 0,1,2, \ldots, x, t \in [0,1]$, So the Banach contraction method achieves convergence.

Table 1: Explain absolute error and mean square error using BCM when $t = 0.1$ and $q = 3$

x	Exact Solution	BCM	EX-BCM
0.1	0.09900	0.09900	$0.10832e^{-11}$
0.2	0.19604	0.19604	$0.16367e^{-11}$
0.3	0.29113	0.29113	$0.16552e^{-11}$
0.4	0.38432	0.38432	$0.16739e^{-11}$
0.5	0.47561	0.47561	$016739e^{-11}$
0.6	0.56506	0.56506	$0.16926e^{-11}$
0.7	0.65268	0.65268	$0.17112e^{-11}$
0.8	0.73849	0.73849	$0.17488e^{-11}$
0.9	0.82254	0.82254	$0.17769e^{-11}$
MSE			$0.27103e^{-23}$

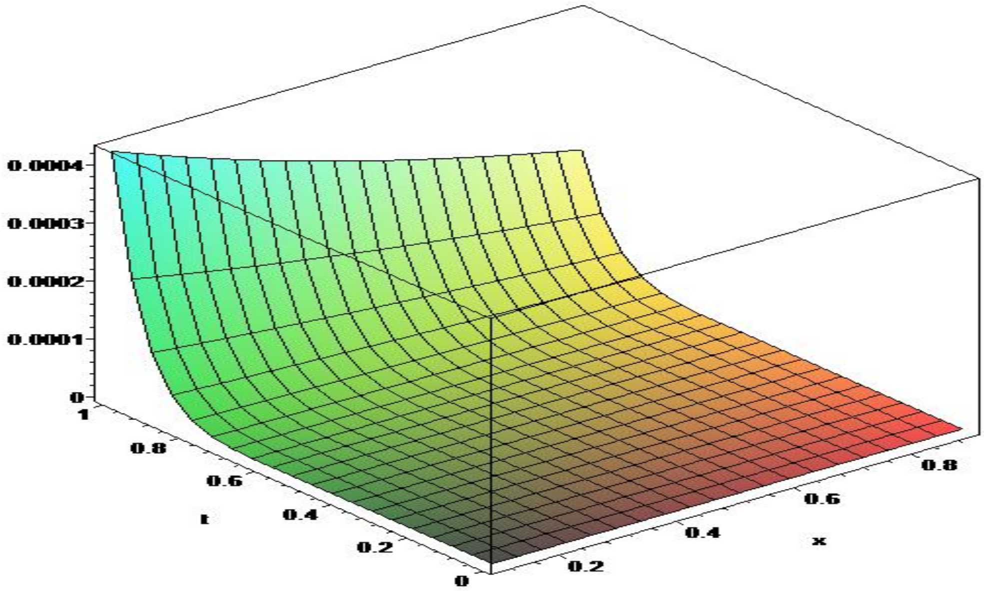

Figure (1): Explains maximum errors at different time and space of the Banach contraction method for q=3.

We notice from Table (1) and Figure (1) that the approximate solution quickly approaches the exact solution, bringing the square error to $0.27103e^{-23}$.

Example 2:

Let us have the following linear partial Integro-differential equations:

$$y_t(x,t) = y_{xx}(x,t) - \int_0^t y(x,s)\, ds + \cos(x) + \sin(x+t) \quad (30)$$

with the initial and boundary conditions as:

$y(x,0) = \sin(x) \quad y(0,t) = \sin(t) \ , \ y(1,t) = \sin(1+t) \qquad x,t \in [0,1]$

The exact solution is $z_{Exact}(x,t) = \sin(x+t)$.

Using the technique described in Part 5 we will obtain the following iterations:

$y_0 = \sin(x) + \cos(x) + t\cos(x) - \cos(x)\cos(t) + \sin(x)\sin(t) + \cdots$

$y_1 = 2\cos(x) - 2\cos(x)\cos(t) + 2\sin(x)\sin(t) - 2t\sin(x) - t^2\cos(x) + \cdots$

$y_2 = -2\sin(x) + 2\cos(x) - 2t\cos(x) - 2\cos(x)\cos(t) + 2\sin(x)\sin(t) + 3\cos(x)\sin(t) + \cdots$

$y_3 = -4\sin(x) - 4t\cos(x) + 5\sin(x)\cos(t) + 5\sin(t)\cos(x) + 2t^2\sin(x) + \frac{2}{3}t^2\cos(x) + \dots$

To prove the convergence, we follow the same steps as the example 1, as shown in Section 4, and thus we obtain values for the convergence parameters as follows:

$$\lambda_0 = \frac{\|z_1(x,t)\|}{\|z_0(x,t)\|} = 0.00994 < 1$$

$$\lambda_1 = \frac{\|z_2(x,t)\|}{\|z_1(x,t)\|} = 0.5635 < 1$$

$$\lambda_2 = \frac{\|z_3(x,t)\|}{\|z_2(x,t)\|} = 0.000029 < 1$$

Since all values of λ_c are less than one $\forall c = 0,1,2, \dots, x, t \in [0,1]$, So the Banach contraction method achieves convergence.

Table 2: Explain absolute error and mean square error using BCM when $t = 0.01$ and $q = 3$.

x	Exact Solution	BCM	EX-BCM
0	0.09983	0.00999	$8.83889e^{-12}$
0.1	0.19867	0.10978	$0.42765e^{-10}$
0.2	0.29552	0.20846	$0.84265e^{-10}$
0.3	0.38941	0.30506	$0.12492e^{-9}$
0.4	0.47943	0.39861	$0.16433e^{-9}$
0.5	0.56462	0.48818	$0.20209e^{-9}$
0.6	0.64422	0.57287	$0.23785e^{-9}$
0.7	0.71736	0.65183	$0.27122e^{-9}$
0.8	0.78333	0.72429	$0.30188e^{-9}$
0.9	0.84147	0.78950	$0.32952e^{-9}$
1.0	0.89121	0.84683	$0.35388e^{-9}$
MSE			$0.49769e^{-19}$

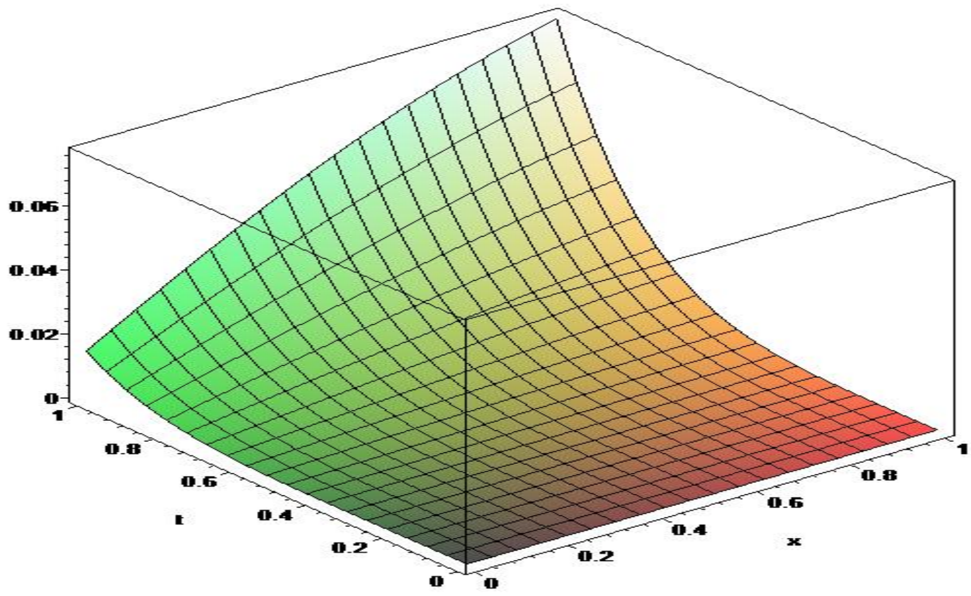

Figure (2): Explains maximum errors at different time and space of the Banach contraction method when q=3 .

We notice from Table (2) and Figure (2) The approximate solution approaches the exact solution as the square error reaches $0.49769e^{-19}$.

Example 3:

Let us have the following nonlinear partial Integro-differential equations:

$$y_t(x,t) = y_{xx}(x,t) + \int_0^t y(x,s)y_x(x,s)\mathrm{ds} - 0.5\left(e^{2(x+t)} - e^{2x}\right) \tag{31}$$

with the initial and boundary conditions as:

$y(x,0) = e^x \quad y(0,t) = e^t, \ y(1,t) = e^{1+t} \qquad x,t \in [0,1]$

The exact solution is $z_{Exact}(x,t) = e^{x+t}$.

Using the technique described in Part 5 we will obtain the following iterations:

$y_0 = e^x + 0.25e^{2x} + 0.5te^{2x} - 0.25e^{2x}e^{2t}$

$y_1 = e^x + 0.75e^{2x} + 1.5te^{2x} - 0.75e^{2x}e^{2t} + t\,e^x + 1.5t^2\,e^{2x} + \ldots$

$y_2 = 2.875t^3e^{3x} + 2.1875t^4e^{4x} - 0.5781e^{4x}e^{2t} + \cdots$

$y_3 = 0.09086t^{10}e^{11x} + 0.0440te^{11x}t^{10}e^{2t} + 5.38523t^8e^{13x}e^{2t} + \cdots.$

$\vdots$

To prove the convergence, we follow the same steps as the example 1, as shown in Section 4, and thus we obtain values for the convergence parameters as follows:

$$\lambda_0 = \frac{\|z_1(x,t)\|}{\|z_0(x,t)\|} = 0.00989 < 1$$

$$\lambda_1 = \frac{\|z_2(x,t)\|}{\|z_1(x,t)\|} = 0.63524 < 1$$

$$\lambda_2 = \frac{\|z_3(x,t)\|}{\|z_2(x,t)\|} = 0.00022 < 1$$

Since all values of λ_c are less than one $\forall c = 0,1,2, \ldots, x, t \in [0,1]$, So the Banach contraction method achieves convergence.

Table 3: Explain absolute error and mean square error using (BCM) when $t = 0.01$ and $q = 3$.

X	Exact Solution	BCM	EX-BCM
0	0.10517	1.01005	$0.16909e^{-6}$
0.1	1.22140	1.11628	$0.20658e^{-6}$
0.2	1.34986	1.23368	$0.25237e^{-6}$
0.3	1.49182	1.36343	$0.30830e^{-6}$
0.4	1.64872	1.50681	$0.37662e^{-6}$
0.5	1.82212	166529	$0.46009e^{-6}$
0.6	2.01375	1.84043	$0.56204e^{-6}$
0.7	2.22554	2.03399	$0.68657e^{-6}$
0.8	2.45960	2.24791	$0.83868e^{-6}$
0.9	2.71828	2.48432	$0.10245e^{-5}$
1	3.00417	2.74560	$0.12514e^{-5}$
MSE			$0.42635e^{-12}$

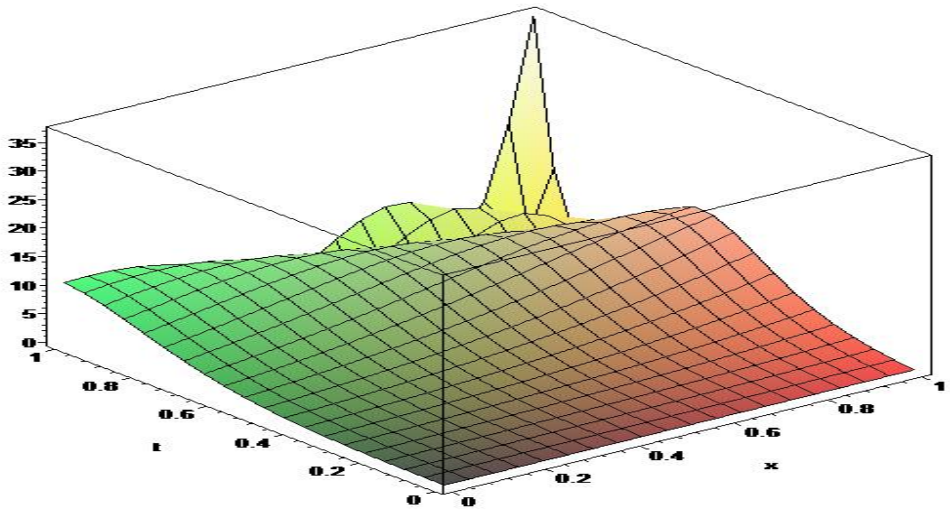

Figure (3): Explains maximum errors at different time and space of the Banach contraction method when q=3 .

We notice from Table (3) and Figure (3) The approximate solution approaches the exact solution as the square error reaches $0.42635e^{-12}$

Example 4[27]:

Let us have the following linear partial Integro-differential equations:

$$y_t(x,t) = y_{xx}(x,t) + (1-x^2)e^t + \frac{(x^2-1)\left(x\cos(xt)+\sin(xt)-xe^t\right)}{(x^2+1)} + 2e^t - \int_0^t \sin\left(x(t-s)\right)y(x,s)\,ds \qquad (32)$$

with the initial and boundary conditions as:

$$y(x,0) = (1-x^2) \quad y(-1,t) = 0, \ y(1,t) = 0 \qquad x,t \in [0,1]$$

The exact solution is $z_{Exact}(x,t) = (1-x^2)e^t$.

Using the technique described in Part 5 we will obtain the following iterations:

$$y_0 = 1 - x^2 + \frac{\cos(xt)-1}{(x^2+1)x} + \frac{x^4}{x^2+1} - \frac{3}{x^2+1} + \frac{x^3}{x^2+1} + \cdots$$

$$y_1 = 1 + \frac{2\cos(xt)}{3} + \frac{\cos(xt)}{x(x^2+1)} - \frac{t^2\sin(xt)}{3} - \frac{4t\cos(xt)}{3} - 6t\cos(xt) + \cdots$$

$$y_2 = 1 - \frac{t^2\sin(xt)}{4(x^6+3x^4+3x^2+1)(x^2+1)} - \frac{t^3\sin(xt)}{6x^3(x^2+1)(x^6+3x^4+3x^2+1)} + \frac{31t\sin(xt)}{4x^5(x^2+1)(x^6+3x^4+3x^2+1)} - \frac{5x^5t^2\sin(xt)}{(x^2+1)(x^6+3x^4+3x^2+1)} - \frac{t^3\cos(xt)}{x^2(x^2+1)(x^6+3x^4+3x^2+1)} + \cdots$$

$$y_3 = 1 - \frac{2029x^8t^3cos(xt)}{12(x^2+1)(x^6+3x^4+3x^2+1)^2} + \frac{350952xt}{(x^2+1)(x^6+3x^4+3x^2+1)^2} + \frac{97720x^3t}{(x^2+1)(x^6+3x^4+3x^2+1)^2} + \frac{633164t}{x(x^2+1)(x^6+3x^4+3x^2+1)^2} + \cdots$$

To prove the convergence, we follow the same steps as the example 1, as shown in Section 4, and thus we obtain values for the convergence parameters as follows:

$$\lambda_0 = \frac{\|z_1(x,t)\|}{\|z_0(x,t)\|} = 0.02442 < 1$$

$$\lambda_1 = \frac{\|z_2(x,t)\|}{\|z_1(x,t)\|} = 0.00134 < 1$$

$$\lambda_2 = \frac{\|z_3(x,t)\|}{\|z_2(x,t)\|} = 0.00046 < 1$$

Since all values of λ_c are less than one $\forall c = 0,1,2, \ldots, x, t \in [0,1]$, So Banach contraction method achieves convergence.

Table 4: Explain absolute error and mean square error using (BCM)

when $t = 0.1$ and $q = 3$

X	Exact Solution	BCM	EX-BCM
0.1	1.09412	1.09412	$0.10019e^{-10}$
0.2	1.06096	1.06096	$0.10003e^{-10}$
0.3	1.00571	1.00570	$0.99866e^{-11}$
0.4	0.92834	0.92834	$0.99695e^{-11}$
0.5	0.82888	0.82888	$0.99518e^{-11}$
0.6	0.70731	0.70731	$0.99333e^{-11}$
0.7	0.56364	0.56364	$0.99142e^{-11}$
0.8	0.39787	0.39786	$0.98943e^{-11}$
0.9	0.20998	0.20998	$0.98737e^{-11}$
1	0	$0.98523e^{-11}$	$0.98523e^{-11}$
MSE			$0.98994e^{-22}$

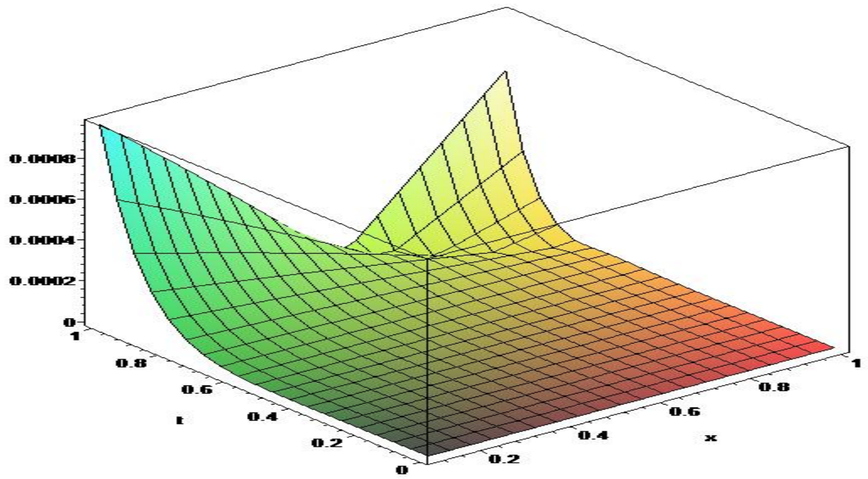

Figure (4): Explains maximum errors at different time and space of the Banach contraction method when q=5 .

We notice from Table (4) and Figure (4) The approximate solution approaches the exact solution as the square error reaches 0.98994_{e}^{-22}.

In this paper, types of Partial integro-differential equations (PIDE) are solved via Banach contraction method and compare the results with the exact solution using absolute error and mean square error (3) and (4), The results in the tables (1-4) show that the Banach contraction method is outstanding in reaching the approximate solution, where the mean square error $MSE = 10^{-23}$. In addition, we note from the tables (1-4) that the mean square error decreases as the value of N increases, that is, the capacity of the iterations used in the numerical solution increases.

6. Conclusions

Partial integro-differential equations (PIDE) occur in many fields of science and mathematics. in this paper, the Banach contraction method used in solving this type from equations. It is an effective method for solving various nonlinear functional equations including partial differential equations (PDEs), ordinary differential equations (ODEs), integral equations, fractional differential equations (FDEs), system of ODEs/FDEs, algebraic equations, in addition to linear and nonlinear integro differential equations. As we can see from Tables (1-4) and Figures (1-4), the approximate solutions using the Banach contraction method quickly approach the exact solutions, as absolute error and square error were relied upon as a measure of error It was noted that the mean square error decreases when q→∞. The Maple program was used in this research to solve the examples, and it is one of the effective programs in solving this type of problem.

Acknowledgement

The authors thank the University of Mosul / College of Computer Science and Mathematics for their support of this paper.

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