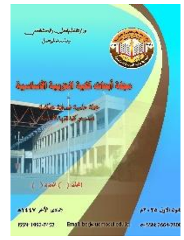
College of Basic Education Researchers Journal, Volume (21) Issue (4) December 2025

College of Basic Education Researches Journal

https://berj.uomosul.edu.iq/



On Strongly γ- Regular Rings

Luma Ahmed Khaleel **Beyda S. Abdullah**
University of Mosul, Department of Mathematical, College of Education for Pure Sciences, Mosul. Iraq.

Article Information

Article history:
Received: August 1,2025
Reviewer: November 19, 2025
Accepted: November 19, 2025
Available online

Keywords:
Strongly Regular Ring, Strongly γ-Regular, GP-Injective Modules.

Correspondence:
E-mail:
l.a.khaleel81@uomosul.edu.iq

Abstract

In our study we examine definition of strongly γ -regular rings and associates, investigate interplay between strongly γ - regular rings and other reduced rings. We also study GP – injective modules, and discuss its relation with strongly γ-regular rings. some important results are secured. Using the connotation of strongly γ-regular rings . researchers concluded:

1- If K be a right semi-duo semi-RG-R satisfying condition (⁎) and K /a K be P –injective for every element a of K. Then K is a strongly γ -RG-R.
2- For a ring K is a reduced satisfying condition (⁎) and every maximal ideal of K is a right annihilator, then K is strongly γ -RG-R.
3- Let K be a ring satisfying condition (⁎). Then K is strongly γ - RG-K if every right K module is GP-injective.
4- Let K be a reduced ring satisfying condition (⁎). Then the center of K is strongly γ - RG-R if K is a right GP-injective ring.

P-ISSN: 1992–7452 **E-ISSN: 2664-2808**

الحلقات المنتظمة القوية من النمط – γ

لمى احمد خليل بيداء سهيل عبد الله

جامعة الموصل، قسم الرياضيات، كلية التربية للعلوم الصرفة، الموصل، العراق.

المستخلص

يهدف البحث الى دراسة تعريف الحلقات المنتظمة القوية من النمط– γ والعلاقة بينها وبين الحلقات الاخرى المختزلة. ودراسة المقاسات الغامرة من النمط– $\mathbb{GP}$ وأوجدنا العلاقة بينها وبين الحلقات المنتظمة القوية من النمط–γ.
وتوصل البحث الى بعض النتائج المهمة بدلالة الحلقات المنتظمة القوية من النمط– γ:

1. إذا كانت $\mathfrak{R}$ حلقة شبه– ديو يمنى لحلقة الشبه المنتظمة تحقق شرط (*) وان $\mathfrak{R}a/\mathfrak{R}$ هي غامرة من النمط P– لكل عنصر a في $\mathfrak{R}$ فان $\mathfrak{R}$ هي حلقة منتظمة قوية من النمط –γ .
2. لتكن $\mathfrak{R}$ حلقة مختزلة تحقق شرط (*) وكل مثالي أعظم في $\mathfrak{R}$ هي تالف ايمن فان $\mathfrak{R}$ هي حلقة منتظمة قوية من النمط –γ .
3. لتكن $\mathfrak{R}$ حلقة تحقق شرط (*) فان $\mathfrak{R}$ هي حلقة منتظمة قوية من النمط –γ إذا كان كل مقاس ايمن هي غامرة من النمط – $\mathbb{GP}$.
4. لتكن $\mathfrak{R}$ حلقة مختزلة تحقق شرط (*) فان مركز $\mathfrak{R}$ هي حلقة منتظمة قوية من النمط– γ إذا كانت $\mathfrak{R}$ حلقة غامرة من النمط – $\mathbb{GP}$.

الكلمات المفتاحية: الحلقة المنتظمة القوية، الحلقة المنتظمة من النمط –γ، المقاسات الغامرة من النمط –$\mathbb{GP}$.

1. Introduction

Rings are algebraic structures that are closed under two binary operations: addition and multiplication. The elements of a ring satisfy specific defining axioms. In this work, $\mathfrak{K}$ denotes an associative ring with unity, and the modules considered are unitary right modules. $\mathfrak{K}$ - .The symbol RG-R will stand the regular ring. The notion of Von Neumann RG-Rs was initially presented by Von Neumann in 1936.(Von Neumann,1936) , called is $RG-R$ (resp. strongly RG-R) if for every $a \in \mathfrak{K}$, has an inner inverse ƀ $\in \mathfrak{K}$ in the sense that a=a ƀ a . (resp. $a = a^2$ ƀ $=$ ƀ a^2).See (Kim, et. al,2024) and (Muhammad& Wahyuni,2023).In (Al-Kouri,1996), gave the present definition of $\pi - RG - R$ is a ring if for every $a \in \mathfrak{K}$, there is an n where $a^n = a^n$ ƀ a^n is regular. Following (Mohammad & Salih,2006), introduced the concept of γ - RG-Rs and showed that any $RG - Rs$ is $\gamma - RG - Rs$ also showed the relation between $\gamma - RG - Rs$ and strongly RG-Rs. Some of the important achievements on certain properties of $\mathfrak{K}$ is a right (left) semi-duo, iff, any principal right (left) ideal of $\mathfrak{K}$ is a two-sided ideal generated by the same element (Yu,1995). (Azumya,1954) Strongly $\pi - RG - R$ is a ring $\mathfrak{K}$ in which every $a \in \mathfrak{K}$, there is ƀ $\in \mathfrak{K}$ and a positive integer n where $a^n = a^{n+1}$ ƀ. For more details see (Chen, et al , 2014), (Danchev,2024), (Nandakumar, et al, 2019) and (Wardayani, et al, 2020). A ring $\mathfrak{K}$ is ĘRT- ring (Ibrahim, 2004) if for each essential right ideal of $\mathfrak{K}$, there is two – sided. Following (Shuker,1994), a ring $\mathfrak{K}$ is a right (left) $semi - RG - R$ if $\forall a \in R$, there is ƀ $\in \mathfrak{K}$ where $a = a$ ƀ (ƀ a) and $r(a) = r$(ƀ) [$l(a) = l$(ƀ)]. We extend the notion of a right $semi - RG - R$ to a right $semi - \pi RG - R$ a ring $\mathfrak{K}$ is right $semi\, \pi - RG$ if and only if any $a \in \mathfrak{K}$ there is a positive integer n and an elements ƀ in $\mathfrak{K}$ where $a^n = a^n$ ƀ and $r(a^n) = r$(ƀ) (Shuker,1994) . $\mathfrak{K}$ is semi-prime if it contains no non zero nilpotent ideals . A ring $\mathfrak{R}$ is said to be a semi-commutative ring if whenever a, ƀ $\in \mathfrak{K}$ such that a ƀ $= 0$, then a). Salih,2006& Mohammad commutative ring.(-.Every reduced ring is semi= 0ƀ $\mathfrak{K}$ respectively. $r(a)$ and $l(a)$ts right and left annihilators are denoted by $\mathfrak{K}^* = \mathfrak{K} \setminus \{0\}$ This paper consists of three items dealing in the second item study where $\mathfrak{K}$ strongly $\gamma - RG$ satisfying (∗) condition and we get relationship between this rings and other ring. In third we discuss a $\mathbb{GP}$ – injective when is $\mathfrak{K}$ strongly γ - RG-R.

2. Some result in Strongly γ-Regular Ring

Definition 2.1 (AL- Hisso,2009)

A ring $\mathfrak{R}$ is called strongly $\gamma - RG - R$ if $\forall a \in \mathfrak{R}$ there is b in $\mathfrak{R}$ sense that $a = a^2 b^n$ where n is a positive integer.

A ring $\mathfrak{R}$ is called strongly $\gamma - RG - R$ if every element of $\mathfrak{R}$ is strongly γ -regular element. For a strongly $\gamma - RG - R$ $\mathfrak{R}$, then one may choose $a = a^2 b^n$ and one has, $a = a^2 b^n = b^n a^2$.

Every strongly $\gamma - RG - R$ is a strongly $RG - R$, however, the converse is not generally true, for examples the ring $(\mathbb{Q}, +, .)$ of rational numbers, the rational (real) and a quadratic field are strongly regulars but not strongly γ -regulars.

Definition 2.2. (Mohammad & Salih,2006**)**

We say that a ring $\mathfrak{R}$ satisfies condition $(*)$ if every $1 \neq a \in \mathfrak{R}$ and $b \in \mathfrak{R}$, sense that $ab = b^m a$,where $m > 1$ is a positive integer . Therefor every quasi-associative ring if satisfying condition $(*)$.

Theorem 2.3. Let $\mathfrak{R}$ be a reduced ring that satisfies condition $(*)$.Then the following statements are equivalent:

1. $\mathfrak{R}$ is strongly $\gamma - RG - R$.
2. $\mathfrak{R}$ is strongly $\pi - RG - R$
3. is $\pi - RG - R$.

Proof.

1⟹2: Since $\mathfrak{R}$ is strongly $\gamma - RG - R$, then by [(Mohammad & Salih,2006); Theorem 5.6] $\mathfrak{R}$ is $\gamma - RG - R$, and since $\mathfrak{R}$ satisfying condition $(*)$ then by [(Mohammad & Salih,2006); Theorem 4.6] R is $RG - R$, and since $\mathfrak{R}$ is reduced, then by (1) Proposition (2.2.4) $\mathfrak{R}$ is strongly $\pi - RG - R$.

2⟹1: Since $\mathfrak{R}$ is a strongly $\pi - RG - R$, then for every $a \in \mathfrak{R}$, there exists $b \in \mathfrak{R}$ and a positive integer n such that $a^n = a^{n+1} b$. implies that $a^n(1 - ab) = 0$. Then $(1 - ab) \in r(a^n) = r(a)$ [(Al-Kouri,1996); Lemma (2.1.9)] $a(1 - ab) = 0$.

Hence $a = a^2 b$. Therefore $\mathfrak{K}$ is strongly $RG-R$. Since $\mathfrak{K}$ satisfying condition (*) then by [(Mohammad & Salih,2006), Theorem. 5.4] $\mathfrak{K}$ is strongly $\gamma - RG-R$.

1⟹3: Since $\mathfrak{K}$ is strongly $\gamma - RG-R$ satisfying condition (*) ,then R is strongly $\pi - RG-R$, then by [(Al-Kouri,1996); Corollary (2.2.7) $\mathfrak{K}$ is $\pi - RG-R$.

3⟹1 Trivial.

Theorem 2.4. Let $\mathfrak{K}$ be a reduced ring satisfying condition (*) then the following are equivalent:

1. $\mathfrak{K}$ is strongly $\gamma - RG-R$.
2. is a right $semi-RG-R$.

Proof.

1⟾2 Let $\mathfrak{K}$ be strongly $\gamma - RG-R$. By [(Mohammad & Salih,2006), Theorem 5.6] $\mathfrak{K}$ is $\gamma - RG-R$, and since R satisfying condition (*), by [(Mohammad & Salih,2006) ,Theorem 4.6] $\mathfrak{K}$ is RG-R. In [(Shuker,1994) , Lemma 2.3] $\mathfrak{K}$ is a right $semi-RG-R$.

2⟾1 Since $\mathfrak{K}$ is a right $semi-RG-R$, As in [(Shuker,1994),Theorem 3.2] r(a) is direct summand for all a in $\mathfrak{K}$, and since $\mathfrak{K}$ satisfying condition (*), so $\mathfrak{K}$ is $\gamma - RG-R$, and since $\mathfrak{K}$ is reduced, so [(Mohammad & Salih,2006), Theorem 5.7] $\mathfrak{K}$ is strongly $\gamma - RG-R$.

Theorem 2.5. Let $\mathfrak{K}$ be a left semi-duo ring satisfying condition (*). Then $\mathfrak{K}$ is a strongly $\gamma - RG-R$ if for any $a \in \mathfrak{K}$ then $a^n R$ is a right semi-regular ideal , where n is a positive integer .

Proof.

Suppose that $a \in \mathfrak{K}$, let n be a positive integer such that $a^n R$ is a right semi-regular ideal, then there exists $y \in a^n \mathfrak{K}$ such that $a^n = a^n y$ and $r(a^n) = r(y)$. Since $\mathfrak{K}$ is a left semi-duo ring, so $a^n \mathfrak{K} = \mathfrak{K} a^n$. But $y \in a^n \mathfrak{K}$, then $y \in \mathfrak{K} a^n$, hence $y = a^n z = b a^n$ for some $z, b \in \mathfrak{K}$. Therefore $a^n = a^n b a^n$. Thus $\mathfrak{K}$ is $\pi - RG-R$. Since . $\gamma - RG-R$is strongly $\mathfrak{K}$satisfying condition (*), in [Theorem 2.3] $\mathfrak{R}$

Theorem 2.6. If $\mathfrak{K}$ be a right semi-duo $semi-RG-R$ satisfying condition (*) and $\mathfrak{K} /a\mathfrak{K}$ be P –injective for every element a of $\mathfrak{K}$. Then $\mathfrak{K}$ is a strongly $\gamma - RG-R$.

Proof.

Since $\mathfrak{K}$ is a right semi-duo semi $-$ RG $-$ R and $\mathfrak{K}/a\mathfrak{K}$ is P $-$ injective for all $a \in \mathfrak{K}$, and [(Shuker,1994),Corollary 2.6] we have $\mathfrak{K}$ is strongly RG $-$ R and since $\mathfrak{K}$ satisfying condition (*) then in view of [(Mohammad & Salih,2006),Theorem 5.4] $\mathfrak{K}$ is strongly γ $-$ RG $-$ R.

Theorem 2.7. Let $\mathfrak{K}$ be reduced satisfying condition (*), if Ƥ a special prime ideal of a ring $\mathfrak{K}$, and if $\mathfrak{K}$ / Ƥ is a right semi-regular. Then $\mathfrak{K}$ is strongly γ $-$ RG $-$ R.

Proof.

Since Ƥ is a special prime ideal of a reduced ring $\mathfrak{K}$ and $\mathfrak{K}$ / Ƥ is a right semi-regular thus, owing to [(Shuker,1994), Theorem 3.2] $\mathfrak{K}$ is a right semi $-$ RG $-$ R, and since $\mathfrak{K}$ satisfying condition (*) then from [Theorem 2.4] $\mathfrak{K}$ is strongly γ $-$ RG $-$ R.

Theorem 2.8. For a ring $\mathfrak{K}$ is a reduced satisfying condition (*) and every maximal ideal of $\mathfrak{K}$ is a right annihilator, then $\mathfrak{K}$ is strongly γ $-$ RG $-$ R.

Proof.

Let $a \in \mathfrak{K}$, we shall prove that $a^n \mathfrak{K} + r(a^n) = \mathfrak{K}$. If not, there exists a maximal right ideal $\mathcal{M}$ containing $a^n R + r(a^n)$. If $\mathcal{M} = r(b)$ for some $0 \neq \mathfrak{K}$, we have $b \in l[a^n \mathfrak{K} + r(a^n)] \subseteq l(a^n) = r(a^n)$, which implies $b \in \mathcal{M} = r(b)$, then $b^2 = 0$ and $b = 0$, a contradiction. Therefore $a^n + r\ \mathfrak{K}$ $(a^n) = \mathfrak{K}$. In particular $a^n c + d = 1$, with $c \in \mathfrak{K}$ and $d \in r(a^n)$, then $a^n c a^n = a^n$, which proves $\mathfrak{K}$ is $\pi -$ RG $-$ R, and since $\mathfrak{K}$ satisfying condition (*)in view of [Theorem 2.3] $\mathfrak{K}$ is strongly γ $-$ RG $-$ R.

Theorem 2.9. For a ring $\mathfrak{K}$ be a duo rings satisfying condition (*). Then $\mathfrak{K}$ is strongly γ $-$ RG $-$ R if for any $a \in \mathfrak{K}$ then $a^n = u$ e for some unites $u \in \mathfrak{K}$ and some idempotent e∈.$\mathfrak{K}$

Proof.

Let a be an element in $\mathfrak{K}$, and let $a^n = u e$ for some unit $u \in \mathfrak{K}$ and some idempotent e∈, but $^n a^1$-u ^{n}e= a ^{n}. auis the inverse of u^{-1}, where $a^{n1-}e = u$. Hence $\mathfrak{K}$ a^n=u e so a^n e = ue. e =ue =a^n. Hence $a^n = a^n u^{-1} a^n$ .Thus $\mathfrak{K}$ is π-RG-R, and since $\mathfrak{K}$ satisfying condition (*) Therefore, with the aid of [Theorem 2.3] .$\mathfrak{K}$ is strongly γ $-$ RG $-$ R

Theorem 2.10. Let $\mathfrak{K}$ be an ȨRT- ring satisfying condition (*). Then $\mathfrak{K}$ is strongly γ $-$ RG $-$ R if $\mathfrak{K}$ is a fully right idempotent.

Proof.

For all $\mathfrak{b} \in \mathfrak{K}$, then $\mathfrak{K} = \mathfrak{b}\mathfrak{K} + s$ is essential in $\mathfrak{K}$ with some right ideal s. Now since $\mathfrak{b} \in (\mathfrak{K}\,\mathfrak{b}\,\mathfrak{K})\mathfrak{b} \subseteq \mathfrak{K}\mathfrak{b}$, then we have $\mathfrak{b} = (\mathfrak{b}a + s)\mathfrak{b}$, for some $a \in \mathfrak{K}$ and $s \in K$. then $\mathfrak{b} - \mathfrak{b}\,a\,\mathfrak{b} = s\,\mathfrak{b} \in s \cap \mathfrak{K}\mathfrak{b} = \{0\}$. Therefore $\mathfrak{b} = \mathfrak{b}\,a\,\mathfrak{b}$, and hence $\mathfrak{K}$ is regular. Since $\mathfrak{K}$ satisfying condition (∗)in view of [(Mohammad & Salih,2006), Theorem 5.5] $\mathfrak{K}$ is strongly $\boldsymbol{\gamma} - RG - R$.

Remark 2.11.

Every strongly $\boldsymbol{\gamma} - RG - R$ is $\boldsymbol{\gamma} - RG - R$, However, the converse does not hold in general. For instance, consider the ring $R_{2\times2}$ (Z) of 2x2 matrices over that ring Z_2 is $\boldsymbol{\gamma} - RG - R$ but not strongly $\boldsymbol{\gamma} - RG - R$ because the element $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ is $\boldsymbol{\gamma}$ − regular element but not strongly $\boldsymbol{\gamma}$ –regular element.

3. Relation between Strongly γ − regular ring and $\mathbb{GP}$ -Injective

In this section we investigate the relatedness between right $\mathbb{GP}$ -injective with strongly $\gamma - RG - R$.

Definition 3.1

A right $\mathfrak{K}$–module $\mathcal{M}$ is referred to as generalized$\boldsymbol{P}$-injective(abbreviated as $\mathbb{GP}$-injective) if for any element in $\mathfrak{K}$, where a positive integer n then $a^n \neq 0$ and every right $\mathfrak{K}$ - homomorphism of $a^n\,\mathfrak{K}$ *at* $\mathcal{M}$ extends to one of $\mathfrak{K}$ into $\mathcal{M}$. For more details see (Chen, et al,2005) and [(Kim& Lee,2011), (Abed, et al, 2022)].

Theorem 3.2. If $\mathfrak{K}$ be a reduced ring where satisfying condition (∗) for any maximal right ideal is $\mathbb{GP} - injective$ Then is an *strongly* $\boldsymbol{\gamma} - RG - R$.

Proof.

Let $a \in \mathfrak{K}$. We Assume that $a^n\,\mathfrak{K} + r(a^n) = \mathfrak{K}$. If not, A maximal right ideal $\mathcal{M}$ exists containing $a^n\,\mathfrak{K} + r(a^n)$. Let $F: a^n\,\mathfrak{K} \longrightarrow \mathcal{M}$ be canonical injective define by $F(a^n\,\mathfrak{b}) = a^n\mathfrak{b}$ for every $\mathfrak{b} \in \mathfrak{K}$. Since $\mathcal{M}$ is $\mathbb{GP} - injective$ then there exists $s \in \mathfrak{K}$ such that $F(a^n\,\mathfrak{b}) = s\,a^n\,\mathfrak{b}$.Therefore $a^n = F(a^n) = sa^n$.Thus $1 - s \in \mathfrak{l}(a^n) = r(a^n) \subseteq \mathcal{M}$, which implies $1 \in \mathcal{M}$, a contradiction. Hence $a^n\,\mathfrak{K} + r(a^n) = \mathfrak{K}$. In particular $a^n s + d = 1$ for some $s \in \mathfrak{K}$ and $d \in r(a^n)$, *so* $a^n s\, a^n = a^n$.Thereforesatisfying $\mathfrak{K}$. Since $\mathfrak{K}$ *is* $\pi - RG - R$ condition (∗), then by [Theorem 2.3] $\mathfrak{K}$ *is strongly* $\gamma - RG - R$.

Theorem 3.3. Let $\mathfrak{R}$ be a ring satisfying condition (∗). Then $\mathfrak{R}$ *is strongly* $\boldsymbol{\gamma} - RG - \mathfrak{R}$ if every right $\mathfrak{R}$ module is $\mathbb{GP}$ – *injective*.

Proof.

Let $a^n\,\mathfrak{R}$ is a principal right ideal of $\mathfrak{R}$ and let $a^n\,\mathfrak{R}$ is $\mathbb{GP}$ – *injective*. For any $a \in \mathfrak{R}$, consider the identity mapping $\mathrm{F}: a^n\,\mathfrak{R} \longrightarrow a^n\,\mathfrak{R}$. Since $a^n\,\mathfrak{R}$ is GP-injective, then there exists $s \in a^n\,\mathfrak{R}$ such that $\mathrm{F}\,(a^n \mathfrak{b}) = sa^n \mathfrak{b}$ for all $\mathfrak{b} \in \mathfrak{R}$. Then $a^n = \mathrm{F}\,(a^n) = sa^n$. Since $s \in a^n\,\mathfrak{R}$, then $s = a^n\,\mathfrak{r}$ for some $\mathfrak{r} \in \mathfrak{R}$, and hence $a^n = a^n\,\mathfrak{r}\,a^n$. Thus $\mathfrak{R}\, is\, \pi - RG - R$. Since $\mathfrak{R}$ satisfying condition (∗),by [Theorem 2.3] $\mathfrak{R}$ *is strongly* $\boldsymbol{\gamma} - RG - R$.

Theorem 3.4. Let $\mathfrak{R}$ be a right duo - ring satisfying condition (∗) with any simple right $\mathfrak{R}$ -module is $\mathbb{GP}$ – *injective*. Then $\mathfrak{R}$ is *strongly* $\gamma - RG - R$.

Proof.

Since $\mathfrak{R}$ be a right duo-ring and every simple right $\mathfrak{R}$ -module is $\mathbb{GP}$ – *injective*, in view of [(Shuker & Mahmood,1994), Theorem 2.4.2] $\mathfrak{R}\, is\, \pi - RG - R$, and since $\mathfrak{R}$ Satisfying condition (∗) by [Theorem 2.3] $\mathfrak{R}$ *is strongly* $\gamma - RG - R$.

Theorem 3.5. Let $\mathfrak{R}$ be a ring satisfying condition (∗) and for every a in $\mathfrak{R}$, $\mathfrak{R}\,/\,\mathfrak{r}\,(a^n)$ *is* $\mathbb{GP}$-injective. Then $\mathfrak{R}$ is *strongly* $\boldsymbol{\gamma} - RG - R$.

Proof.

Let a be a non-zero element in R. Define a right $\mathfrak{R}$ -homomorphism $\mathrm{F}: a^n\,\mathfrak{R} \longrightarrow \mathfrak{R}\,/\,\mathfrak{r}\,(a^n)\; by\; \mathrm{F}\,(a^n\,x) = x + \mathfrak{r}\,(a^n)\; for\; all\; x \in \mathfrak{R}$, where F is well defined , $let\; a^n x_1 = a^n x_2$, for any two elements $x_1, x_2 \in \mathfrak{R}$ then $a^n\,(x_1 - x_2) = 0$, so $x_1 - x_2 \in \mathfrak{r}\,(a^n)$ and hence $\mathrm{F}\,(a^n x_1) = x_1 + \mathfrak{r}\,(a^n) = x_2 + \mathfrak{r}\,(a^n) = \mathrm{F}\,(a^n\,x_2)$.

Since $\mathfrak{R}\,/\,\mathfrak{r}\,(a^n)$ *is* $\mathbb{GP}$-injective, so if $s \in \mathfrak{R}$ then $\mathrm{F}\,(a^n\,x) = (s + \mathfrak{r}\,(a^n))\,a^n x = sa^n x + \mathfrak{r}\,(a^n)$.Now $\mathrm{F}\,(a^n) = 1 + \mathfrak{r}\,(a^n) = s\,a^n + \mathfrak{r}\,(a^n)$, which implies $1 - sa^n \in \mathfrak{r}\,(a^n)$ $.So\; a^n = a^n s\, a^n$. Hence $is\; \pi - RG - R$. If $\mathfrak{R}$ satisfying condition (∗), and notice that using [Theorem 2.3] $\mathfrak{R}$ *is strongly* $\gamma - RG - R$.

Theorem 3.6. Let $\mathfrak{R}$ be a reduced ring satisfying condition (∗). Then the center of $\mathfrak{R}$ is strongly γ - RG-R if $\mathfrak{R}$ is a right $\mathbb{GP}$-injective ring.

Proof.

Let C the center of $\mathfrak{R}\,, 0 \neq s \in C\; and\; let\; u \in \ell(\mathfrak{r}\,(s^n)) = \ell\,(\mathfrak{r}\,(\mathfrak{R}\,s^n)).\, Define\; \mathrm{F}: s^n\,\mathfrak{R} \longrightarrow \mathfrak{R}\; is\; a$ right $\mathfrak{R}$ –

homomorphism by $F(s^n x) = u x \,\forall x \in \mathfrak{K}$, then F is well -defined, indeed, let $s^n x_1 = s^n x_2$ for any two elements x_1, x_2 in $\mathfrak{K}$, then $s^n (x_1 - x_2) = 0$, so $x_1 - x_2 \in r(s^n)$.Since $r(s^n) = r(\ell(r(s^n))) \in r(u)$, then $x_1 - x_2 \in r(u)$ implies $u(x_1 - x_2) = 0$ Hence $ux_1 = ux_2$. Therefore $F(s^n x_1) = ux_1 = ux_2 = F(s^n x_2)$. Now, since $\mathfrak{K}$ is $\mathbb{GP}$-injective, then there exists $b \in \mathfrak{K}$ such that $ux = F(s^n x) = bs^n x$. now, $u = F(s^n) = b s^n \in \mathfrak{K} s^n$. Thus $\ell\,[(r(s^n)] = \mathfrak{K} s^n$. Since $r(s^n) \subseteq r(s^{n+1})$, then $s^n = d s^{n+1}$ for some $d \in \mathfrak{K}$, therefore s is strongly $\pi - RG - R$, and since $\mathfrak{K}$ satisfying condition (∗) and notice that using [Theorem 2.3] s is strongly $\gamma - RG - R$.

4. Conclusion

Our aim in this work is to study strongly γ - regular rings when ring $\mathfrak{K}$ is associative ring with unity and clarify some important properties. Finally study $\mathbb{GP}$ – injective and its relationship with strongly γ -regular rings.

5. References

Al-Kouri, M. R. M. (1996). "On π-Regular Rings" (Doctoral dissertation, M. Sc. Thesis, Mosul University).

AL- Hisso Sh, S. E . (2009). On Some Types of Strongly Regular Rings, M. Sc. Thesis, Mosul University.

Azumya,G. (1954). Strongly π-regular ring, J. Fac. Sci. Hokkaido Univ. Series I. Mathematics, 13(1), 34-39.

Chen,H. Kose,H.and Kurtulmaz ,Y. (2014). Extensions of strongly π-regular rings ,Bull. Korean Math. Soc. 51, 555–565.

Chen, J. , Zhou, X. and Zhu, Z. (2005). Gp – injective rings need not be P-injective , comm. Algebra, 33 , 2395 - 2402 .

Danchev, P. V. (2024). A symmetric generalization of π-regular rings. Ricerche di Matematica, 73(1), 179-190.

Ibrahim, Z.M . (2004). On ERT and MERT – rings . AL-Rafidain Journal of Computer Sciences and Mathematics, 1(1), 28-33.

Kim, N. K., Kwak, T. K., Lee, Y, & Ryu, S. J. (2024). On von Neumann Regularity of Commutators. In Algebra Colloquium (Vol. 31, No. 02, pp. 181-198). World Scientific Publishing Company.

Kim, N. K., & Lee, Y. (2011). On strong π-regularity and π-regularity. Communications in Algebra, 39(11), 4470-4485.

Abed, M. M., Al-Sharqi, F., & Zail, S. H. (2022). Injective modules, projective modules and their relationships to some rings. In AIP Conference Proceedings (Vol. 2472, No. 1). AIP Publishing.

Mohammad, A . J. and Salih, S.M. (2006). On γ -regular rings, J. Edu. Sci, 18(4).

Muhammad, H., & Wahyuni, S. (2023). Generalization of von-neumann regular rings to von-neumann regular modules. Barekeng: Jurnal Ilmu Matematika dan Terapan, 17(4), 1885-1892.

Von Neumann, J. (1936). On Regular rings, Proceeding of the National Academy of Sciences, 22(12), 707-713Acad.

Nandakumar. P, Dheena P .and Kanthakumar ,D. (2019), A Type of Strongly Regular Semi rings, The Journal of Fuzzy Mathematics, 27(2), 487–498.

Shuker, N.H. (1994). On semi-regular rings. Journal of education and science, 21, 183-187.

Shuker, N.H, Mahmood. A.S. (1994). On π-regular ring. Journal of education and science, Vol. (22).

Wardayani, A., Kharismawati, I., Sihwaningrum ,I. (2020). Regular rings and their properties. In Journal of Physics: Conference Series (Vol. 1494, No. 1, p. 012020). IOP Publishing.

Yu ,H.P. (1995) . On quasi-duo rings, Glasgow Mathematical Journal, 37, 21-30.